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Commuting Eulerian operators

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ABSTRACT

Motivated by the work of Visontai and Dey–Sivasubramanian on the gamma-positivity of some polynomials, we discover the commutative property of a pair of Eulerian operators. As an application, we show the bi-gamma-positivity of the descent polynomials on permutations of the multiset $\{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\}$, where $0 \leqslant a_i \leqslant 2$. Therefore, these descent polynomials are all alternatingly increasing, and so they are unimodal with modes in the middle.

1. Introduction

Throughout this paper, we always let $f(x) = \sum_{i=0}^{n} f_i x^i$ be a polynomial with nonnegative coefficients. We say that $f(x)$ is *unimodal* if $f_0 \leqslant f_1 \leqslant \cdots \leqslant f_k \geqslant f_{k+1} \geqslant \cdots \geqslant f_n$ for some k, where the index k is called the *mode* of $f(x)$. It is well known that if $f(x)$ has only nonpositive real zeros, then $f(x)$ is unimodal (see [5, p. 419] for instance). If $f(x)$ is symmetric with the center of symmetry $\lfloor n/2 \rfloor$, i.e., $f_i = f_{n-i}$ for all indices $0 \leqslant i \leqslant n$, then it can be expanded as

$$f(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_k x^k (1+x)^{n-2k}.$$

The polynomial $f(x)$ is *γ-positive* if $\gamma_k \geqslant 0$ for all $0 \leqslant k \leqslant \lfloor n/2 \rfloor$. Clearly, γ-positivity implies symmetry and unimodality. Let $f(x,y) = \sum_{i=0}^{n} f_i x^i y^{n-i}$ be a homogeneous bivariate polynomial. We say that $f(x,y)$ is *bivariately γ-positive* with the center of symmetry $\frac{n}{2}$ if $f(x,y)$ can be written as follows:

$$f(x,y) = y^{\delta} \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_k (xy)^k (x+y)^{n-2k},$$

where $\delta = 0$ or $\delta = 1$. There has been considerable recent interest in the study of the γ-positivity of polynomials, see [3,20,26] and references therein. In particular, Brändén [3, Remark 7.3.1] noted that if $f(x)$ is symmetric and has only real zeros, then it is γ-positive.

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Assume that $\deg f(x) = n$. Following [2,4], there is a unique symmetric decomposition $f(x) = a(x) + xb(x)$, where

$$a(x) = \frac{f(x) - x^{n+1}f(1/x)}{1-x},\ \ b(x) = \frac{x^n f(1/x) - f(x)}{1-x}.$$

According to [18, Definition 8], the polynomial $f(x)$ is said to be *bi-γ-positive* if both $a(x)$ and $b(x)$ are γ-positive. It should be noted that γ-positivity is a special case of bi-γ-positivity. Following [22, Definition 2.9], the polynomial $f(x)$ is *alternatingly increasing* if

$$f_0 \leqslant f_n \leqslant f_1 \leqslant f_{n-1} \leqslant \cdots \leqslant f_{\lfloor (n+1)/2 \rfloor}.$$

In recent years, Beck–Jochemko–McCullough [1], Brändén–Solus [4] and Solus [24] studied the alternatingly increasing property of several h^*-polynomials as well as some refined Eulerian polynomials. It is known that the polynomial $f(x)$ is alternatingly increasing if and only if the pair of polynomials in its symmetric decomposition are both unimodal and have only nonnegative coefficients, which was first pointed out by Brändén–Solus [4]. Thus bi-γ-positivity is stronger than alternatingly increasing property.

For $\mathbf{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{N}^n$, we define the multiset $[n]_{\mathbf{m}} = \{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$, where each element i appears m_i times. The set of all permutations of $[n]_{\mathbf{m}}$ is denoted as $\mathfrak{S}_{\mathbf{m}}$. When $m_1 = m_2 = \cdots = m_n = 1$, $\mathfrak{S}_{\mathbf{m}}$ reduces to the symmetric group $\mathfrak{S}_n$, representing the set of all permutations of $\{1, 2, \ldots, n\}$. We refer to a permutation in $\mathfrak{S}_{\mathbf{m}}$ as a *generalized Stirling permutation* if, for each i within the range $1 \le i \le n$, the elements located between two consecutive occurrences of i are greater than i. For more details, refer to [19,20]. In the following, we shall consider Eulerian polynomials of a multiset. Along the same lines, it would be interesting to study some enumerative polynomials of Stirling permutations.

Set $m = \sum_{i=1}^n m_i$. For $\pi = \pi_1\pi_2 \ldots \pi_m \in \mathfrak{S}_{\mathbf{m}}$, we always assume that $\pi_0 = \pi_{m+1} = 0$ (except where explicitly stated). If $i \in \{0, 1, 2, \ldots, m\}$, then π_i is called an *ascent* (resp. a *descent*) if $\pi_i < \pi_{i+1}$ (resp. $\pi_i > \pi_{i+1}$). Let $\mathrm{asc}\,(\pi)$ (resp. $\mathrm{des}\,(\pi)$) be the number of ascents (resp. descents) of π. The *multiset Eulerian polynomials* $A_{\mathbf{m}}(x)$ are defined by

$$A_{\mathbf{m}}(x) = \sum_{\pi\in\mathfrak{S}_{\mathbf{m}}} x^{\mathrm{asc}\,(\pi)} = \sum_{\pi\in\mathfrak{S}_{\mathbf{m}}} x^{\mathrm{des}\,(\pi)}.$$

A classical result of MacMahon [21, Vol 2, Chapter IV, p. 211] says that

$$\frac{A_{\mathbf{m}}(x)}{(1-x)^{1+m}} = \sum_{k\geqslant 0} \binom{k+m_1}{m_1}\binom{k+m_2}{m_2}\cdots\binom{k+m_n}{m_n} x^{k+1}. \tag{1}$$

As usual, we write $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$. Denote by $A_{\pi(\mathbf{m})}(x)$ the descent polynomial on multipermutations of $\{\pi_1^{m_1}, \pi_2^{m_2}, \ldots, \pi_n^{m_n}\}$. It follows from (1) that

$$A_{\mathbf{m}}(x) = A_{\pi(\mathbf{m})}(x). \tag{2}$$

When $\mathbf{m} = (1, 1, \ldots, 1)$, the polynomial $A_{\mathbf{m}}(x)$ is reduced to the classical Eulerian polynomial $A_n(x)$. In other words,

$$A_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{asc}\,(\pi)} = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{des}\,(\pi)}.$$

Simion [23, Section 2] found that $A_{\mathbf{m}}(x)$ is real-rooted for any $\mathbf{m}$. When $\mathbf{m} = (p, p, \ldots, p)$, Carlitz–Hoggatt [6] showed that $A_{\mathbf{m}}(x)$ is symmetric, where p is a given positive integer. By [3, Remark 7.3.1], an immediate consequence is the following result.

Proposition 1. *For any $\mathbf{m}$, the multiset Eulerian polynomials $A_{\mathbf{m}}(x)$ are all unimodal. When $\mathbf{m} = (p, p, \ldots, p)$, the polynomial $A_{\mathbf{m}}(x)$ is γ-positive, and so its mode is in the middle.*

Recently, there has been much work on enumerative polynomials of multisets, see [13–16,26] for instance. In particular, Lin–Xu–Zhao [15] found a combinatorial interpretation for the γ-coefficients of $A_{\mathbf{m}}(x)$ via a model of weakly increasing trees, where $\mathbf{m} = (p, p, \ldots, p)$. Motivated by Proposition 1, it is natural to consider the following problem.

Problem 2. For any $\mathbf{m}$, could we characterize the location of the mode of $A_{\mathbf{m}}(x)$?

A bivariate version of the Eulerian polynomial over the symmetric group is given as follows:

$$A_n(x, y) = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{asc}\,(\pi)} y^{\mathrm{des}\,(\pi)}.$$

In particular, $A_n(x, 1) = A_n(1, x) = A_n(x)$. Carlitz and Scoville [7] found that

$$A_{n+1}(x, y) = xy\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) A_n(x, y),\ \ A_1(x, y) = xy.$$

Using the following Eulerian operator

$$T = xy\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \tag{3}$$

Foata and Schützenberger [11] discovered that

$$A_n(x,y)=\sum_{k=1}^{\lfloor (n+1)/2\rfloor}\gamma(n,k)(xy)^k(x+y)^{n+1-2k},$$

where $\gamma(n,k)$ are all nonnegative integers. Applying the same idea, Visontai [25] investigated the joint generating polynomial of descents and inverse descents, Dey–Sivasubramanian [9] studied the descent polynomials on permutations in the alternating group. As an illustration, we now recall a result on the Eulerian operator T, which is a slightly variant of [9, Lemma 5].

Lemma 3. *Let $f(x,y)$ be a bivariate γ-positive polynomial with the center of symmetry $\frac{n}{2}$. Then $T(f(x,y))$ is a bivariate γ-positive polynomial with the center of symmetry $\frac{n+1}{2}$.*

Motivated by the work of Visontai [25] and Dey–Sivasubramanian [9], in this paper we introduce the following Eulerian operator

$$G=xy^2\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)+\frac{x^2y^2}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+x^2y^2\frac{\partial^2}{\partial x\partial y}. \tag{4}$$

In the next section, we prove the commutative property of the Eulerian operators T and G. In Section 3, we shall prove the following result, which gives a partial answer to Problem 2.

Theorem 4. *Let $[n]_{\mathbf{m}}=\{1^{m_1},2^{m_2},\ldots,n^{m_n}\}$, where $0\leqslant m_i\leqslant 2$. The Eulerian polynomials $A_{\mathbf{m}}(x)$ are all bi-γ-positive, and so $A_{\mathbf{m}}(x)$ are all alternatingly increasing. The polynomial $A_{\mathbf{m}}(x)$ becomes a γ-positive polynomial only when $\mathbf{m}$ consists exclusively of the elements $\{0,1\}$ or $\{0,2\}$.*

In the following discussion, we always set $\mathbf{m}=\{m_1,m_2,\ldots,m_n\}$, where $0\leqslant m_i\leqslant 2$. Let

$$A_{\mathbf{m}}(x,y)=\sum_{\pi\in\mathfrak{S}_{\mathbf{m}}}x^{\operatorname{des}(\pi)}y^{m+1-\operatorname{des}(\pi)}.$$

where $m=\sum_{i=1}^n m_i$. Clearly, $A_{\mathbf{m}}(x,1)=A_{\mathbf{m}}(x)$. For convenience, set $A_{\emptyset}(x,y)=x$.

Example 5. We have

$$A_{\{1\}}(x,y)=xy,\ \ A_{\{1,1\}}(x,y)=xy(x+y),\ \ A_{\{1,1,1\}}(x,y)=xy(x^2+4xy+y^2),$$

$$A_{\{2\}}(x,y)=xy^2,\ \ A_{\{1,2\}}(x,y)=A_{\{2,1\}}(x,y)=xy^2(2x+y),\ \ A_{\{2,2\}}(x,y)=xy^2(x^2+4xy+y^2),$$

$$A_{\{1,1,2\}}(x,y)=A_{\{1,2,1\}}(x,y)=A_{\{2,1,1\}}(x,y)=xy^2(4x^2+7xy+y^2),$$

$$A_{\{1,2,2\}}(x,y)=A_{\{2,1,2\}}(x,y)=A_{\{2,2,1\}}(x,y)=xy^2(2x^3+15x^2y+12xy^2+y^3).$$

Remark 6. In [17, Section 4], by using context-free grammars, Ma–Ma–Yeh proved that the descent polynomials of all signed permutations of the multiset $\{1^2,2^2,\ldots,n^2\}$ are bi-γ-positive. Based on the work of Brändén–Solus [4] and Lin [12], Ding–Zhu [10, Proposition 5.6] found that the descent polynomials of all signed permutations of the multiset $\{1^{s_1},2^{s_2},\ldots,n^{s_n}\}$ are alternatingly increasing, where $s_j\in\{1,2\}$ for all $1\leqslant j\leqslant n$.

2. The commutative property of Eulerian operators

Lemma 7. *Let $\mathbf{m}=\{m_1,m_2,\ldots,m_n\}$, where $0\leqslant m_i\leqslant 2$. Set $\overline{\mathbf{m}}=\mathbf{m}\cup\{n+1\}$ and $\underline{\mathbf{m}}=\mathbf{m}\cup\{n+1,n+1\}$. Let T and G be the Eulerian operators defined by (3) and (4), respectively. Then we have $A_{\overline{\mathbf{m}}}(x,y)=T\left(A_{\mathbf{m}}(x,y)\right)$ and $A_{\underline{\mathbf{m}}}(x,y)=G\left(A_{\mathbf{m}}(x,y)\right)$.*

Proof. Let $\pi\in\mathfrak{S}_{\mathbf{m}}$. We introduce a labeling of π as follows:

(L_1) if π_i is a descent, then put a superscript label x right after it;
(L_2) if π_i is an ascent or a plateau (i.e., $\pi_i=\pi_{i+1}$), then put a superscript label y right after it.

It should be noted that there always exist a superscript label y before π, and a superscript label x at the end of π, since we always assume that $\pi_0=\pi_{m+1}=0$. For example, for $\pi=12125433$, the labeling of π is given by ${}^y1^y2^x1^y2^y5^x4^x3^y3^x$.

When the element $n+1$ is inserted into π, we always get a label y just before $n+1$ as well as a label x right after $n+1$. This corresponds to the substitution rule of labels: $x\to xy$ or $y\to xy$. By induction, we see that the term $T\left(A_{\mathbf{m}}(x,y)\right)$ gives the contribution of all $\pi'\in\mathfrak{S}_{\overline{\mathbf{m}}}$ in which the element $n+1$ appears in positions j, where $0\leqslant j\leqslant m$. Hence $A_{\overline{\mathbf{m}}}(x,y)=T\left(A_{\mathbf{m}}(x,y)\right)$.

When the two copies of $n+1$ are inserted into π, we distinguish among three distinct cases:

(c_1) If the two consecutive copies of $n+1$ are inserted into a position of π, then the changes of labeling are illustrated as follows:

$$\cdots \pi_i^x \pi_{i+1} \cdots \to \cdots \pi_i^y (n+1)^y (n+1)^x \pi_{i+1} \cdots,$$

$$\cdots \pi_i^y \pi_{i+1} \cdots \to \cdots \pi_i^y (n+1)^y (n+1)^x \pi_{i+1} \cdots.$$

This explains the term $xy^2\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)$;

(c_2) If the two copies of $n+1$ are inserted into two different positions with the same label, then the changes of labeling are illustrated as follows:

$$\cdots \pi_i^x \pi_{i+1} \cdots \pi_j^x \pi_{j+1} \cdots \to \cdots \pi_i^y (n+1)^x \pi_{i+1} \cdots \pi_j^y (n+1)^x \pi_{j+1} \cdots,$$

$$\cdots \pi_i^y \pi_{i+1} \cdots \pi_j^y \pi_{j+1} \cdots \to \cdots \pi_i^y (n+1)^x \pi_{i+1} \cdots \pi_j^y (n+1)^x \pi_{j+1} \cdots.$$

This explains the term $\frac{x^2y^2}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$;

(c_3) If the two copies of $n+1$ are inserted into two different positions with different labels, then the changes of labeling are illustrated as follows:

$$\cdots \pi_i^x \pi_{i+1} \cdots \pi_j^y \pi_{j+1} \cdots \to \cdots \pi_i^y (n+1)^x \pi_{i+1} \cdots \pi_j^y (n+1)^x \pi_{j+1} \cdots,$$

$$\cdots \pi_i^y \pi_{i+1} \cdots \pi_j^x \pi_{j+1} \cdots \to \cdots \pi_i^y (n+1)^x \pi_{i+1} \cdots \pi_j^y (n+1)^x \pi_{j+1} \cdots.$$

This explains the term $x^2y^2\frac{\partial^2}{\partial x \partial y}$.

Therefore, the action of G on the set of labeled multipermutations in $\mathfrak{S}_{\mathbf{m}}$ gives the set of labeled multipermutations in $\mathfrak{S}_{\underline{\mathbf{m}}}$. This yields $A_{\underline{\mathbf{m}}}(x,y)=G\left(A_{\mathbf{m}}(x,y)\right)$. □

We can now present the following result.

Theorem 8. *The Eulerian operators T and G are commutative, i.e., $GT=TG$.*

Proof. Let $G=G_1+G_2+G_3$, where

$$G_1=xy^2\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right),\ G_2=\frac{x^2y^2}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right),\ G_3=x^2y^2\frac{\partial^2}{\partial x\partial y}. \tag{5}$$

It is easily checked that

$$\begin{aligned}
G_1T &= xy^2\left[(x+y)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)+xy\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+2\frac{\partial^2}{\partial x\partial y}\right)\right],\\
G_2T &= \frac{x^2y^2}{2}\left[2y\frac{\partial^2}{\partial x^2}+2x\frac{\partial^2}{\partial y^2}+2(x+y)\frac{\partial^2}{\partial x\partial y}+xy\left(\frac{\partial^3}{\partial x^3}+\frac{\partial^3}{\partial y^3}+\frac{\partial^3}{\partial x^2\partial y}+\frac{\partial^3}{\partial y^2\partial x}\right)\right],\\
G_3T &= x^2y^2\left[\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)+(x+y)\frac{\partial^2}{\partial x\partial y}+x\frac{\partial^2}{\partial x^2}+y\frac{\partial^2}{\partial y^2}+xy\left(\frac{\partial^3}{\partial x^2\partial y}+\frac{\partial^3}{\partial y^2\partial x}\right)\right],\\
TG_1 &= xy\left[(2xy+y^2)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)+xy^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+2\frac{\partial^2}{\partial x\partial y}\right)\right],\\
TG_2 &= xy\left[(xy^2+x^2y)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{x^2y^2}{2}\left(\frac{\partial^3}{\partial x^3}+\frac{\partial^3}{\partial y^3}+\frac{\partial^3}{\partial x^2\partial y}+\frac{\partial^3}{\partial y^2\partial x}\right)\right],\\
TG_3 &= xy\left[2(x^2y+xy^2)\frac{\partial^2}{\partial x\partial y}+x^2y^2\left(\frac{\partial^3}{\partial x^2\partial y}+\frac{\partial^3}{\partial y^2\partial x}\right)\right].
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
GT=TG&=(xy^3+2x^2y^2)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)+(2x^2y^3+x^3y^2)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\\
&(4x^2y^3+2x^3y^2)\frac{\partial^2}{\partial x\partial y}+\frac{x^3y^3}{2}\left(\frac{\partial^3}{\partial x^3}+\frac{\partial^3}{\partial y^3}\right)+\frac{3x^3y^3}{2}\left(\frac{\partial^3}{\partial x^2\partial y}+\frac{\partial^3}{\partial y^2\partial x}\right).
\end{aligned} \tag{6}$$

This completes the proof. □

Example 9. Note that $A_{\{2\}}(x,y) = xy^2$. Using (6), one has

$$GT(xy^2) = A_{\{2,1,2\}}(x,y) = xy^2(2x^3 + 15x^2y + 12xy^2 + y^3) = A_{\{2,2,1\}}(x,y) = TG(xy^2).$$

3. The proof of Theorem 4

We claim that there are three types of bivariate γ-expansions for $A_{\mathbf{m}}(x,y)$.

$$\text{Type } I: \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} a(m,k)(xy)^k(x+y)^{m+1-2k}, \tag{7}$$

$$\text{Type } II: y\sum_{k=1}^{\lfloor m/2 \rfloor} b(m,k)(xy)^k(x+y)^{m-2k}, \tag{8}$$

$$\text{Type } III: \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} c(m,k)(xy)^k(x+y)^{m+1-2k} + y\sum_{k=1}^{\lfloor m/2 \rfloor} d(m,k)(xy)^k(x+y)^{m-2k}, \tag{9}$$

where the first type corresponds to $\mathbf{m} = \{1,1,\ldots,1\}$, the second type corresponds to $\mathbf{m} = \{2,2,\ldots,2\}$, and the last type corresponds to the other cases. As illustrated by Example 5, the claim holds for any $m \leqslant 4$.

We proceed by induction. It suffices to distinguish among three distinct cases:

(a_1) Consider the case $\mathbf{m} = \{1,1,\ldots,1\}$. Suppose that the γ-expansion of $A_{\mathbf{m}}(x,y)$ is given by Type I, i.e., it is bivariate γ-positive with the center of symmetry $\frac{m+1}{2}$. By (7) and Lemma 7, we have

$$\begin{aligned} A_{\overline{\mathbf{m}}}(x,y) &= T\left(A_{\mathbf{m}}(x,y)\right) \\ &= T\left(\sum_{k=1}^{\lfloor (m+1)/2 \rfloor} a(m,k)(xy)^k(x+y)^{m+1-2k}\right) \\ &= \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} a(m,k)[k(xy)^k(x+y)^{m+2-2k} + 2(m+1-2k)(xy)^{k+1}(x+y)^{m-2k}], \end{aligned}$$

Setting $\widetilde{a}(m,k) = ka(m,k) + 2(m+3-2k)a(m,k-1)$, we get

$$A_{\overline{\mathbf{m}}}(x,y) = T\left(A_{\mathbf{m}}(x,y)\right) = \sum_{k=1}^{\lfloor (m+2)/2 \rfloor} \widetilde{a}(m,k)(xy)^k(x+y)^{m+2-2k}. \tag{10}$$

So the γ-expansion of $A_{\overline{\mathbf{m}}}(x,y)$ belongs to Type I.

Consider the action of the operator G on the basis element $(xy)^k(x+y)^{m+1-2k}$. We get

$$G\left((xy)^k(x+y)^{m+1-2k}\right) = G_1\left((xy)^k(x+y)^{m+1-2k}\right) + (G_2+G_3)\left((xy)^k(x+y)^{m+1-2k}\right),$$

where G_1, G_2 and G_3 are respectively defined by (5). After some calculations, this gives the following:

$$G_1\left((xy)^k(x+y)^{m+1-2k}\right) = y\left[k(xy)^k(x+y)^{m+2-2k} + 2(m+1-2k)(xy)^{k+1}(x+y)^{m-2k}\right],$$

$$(G_2+G_3)\left((xy)^k(x+y)^{m+1-2k}\right) = \binom{k}{2}(xy)^k(x+y)^{m+3-2k} + k(xy)^{k+1}(x+y)^{m+3-2(k+1)} +$$
$$2k(m+1-2k)(xy)^{k+1}(x+y)^{m+3-2(k+1)} + 2(m+1-2k)(m-2k)(xy)^{k+2}(x+y)^{m+3-2(k+2)}.$$

Thus $G_1\left((xy)^k(x+y)^{m+1-2k}\right)$ and $(G_2+G_3)\left((xy)^k(x+y)^{m+1-2k}\right)$ are both bivariate γ-positive polynomials with the center of symmetry $\frac{m+2}{2}$ and $\frac{m+3}{2}$, respectively. Therefore, the γ-expansion of $G\left(A_{\mathbf{m}}(x,y)\right)$ belongs to Type III. More precisely, there exist nonnegative integers $c(m,k)$ and $d(m,k)$ such that

$$\begin{aligned} A_{\underline{\mathbf{m}}}(x,y) = G\left(A_{\mathbf{m}}(x,y)\right) = &\sum_{k=1}^{\lfloor (m+1)/2 \rfloor} c(m,k)(xy)^k(x+y)^{m+3-2k} + \\ &y\sum_{k=1}^{\lfloor m/2 \rfloor} d(m,k)(xy)^k(x+y)^{m+2-2k}. \end{aligned} \tag{11}$$

(a_2) Consider the case $\mathbf{m} = \{2,2,\ldots,2\}$. By (8) and Lemma 7, we have

$$T\left(A_{\mathbf{m}}(x,y)\right)$$

$$= T\left(y\sum_{k=1}^{\lfloor m/2\rfloor} b(m,k)(xy)^k(x+y)^{m-2k}\right)$$
$$= \sum_{k=1}^{\lfloor m/2\rfloor} b(m,k)(xy)^{k+1}(x+y)^{m-2k} + yT\left(\sum_{k=1}^{\lfloor m/2\rfloor} b(m,k)(xy)^k(x+y)^{m-2k}\right)$$
$$= \sum_{k=2}^{\lfloor (m+2)/2\rfloor} b(m,k-1)(xy)^k(x+y)^{m+2-2k} +$$
$$y\sum_{k=1}^{\lfloor m/2\rfloor} b(m,k)\left[k(xy)^k(x+y)^{m+1-2k} + 2(m-2k)(xy)^{k+1}(x+y)^{m-2k-1}\right],$$
$$= \sum_{k=2}^{\lfloor (m+2)/2\rfloor} b(m,k-1)(xy)^k(x+y)^{m+2-2k} + y\sum_{k=1}^{\lfloor (m+1)/2\rfloor} \widetilde{b}(m,k)(xy)^k(x+y)^{m+1-2k},$$

where $\widetilde{b}(m,k) = kb(m,k) + 2(m-2k+2)b(m,k-1)$. Thus the γ-expansion of $T(A_{\mathbf{m}}(x,y))$ belongs to the Type *III*. Consider the action of the operator G on the basis element $y(xy)^p(x+y)^q$. After some simplifications, it is routine to verify that $G\left(x^py^{p+1}(x+y)^q\right)$ has the following expansion:

$$y\left[\binom{p+1}{2}(xy)^p(x+y)^{q+2} + (1+p)(1+2q)(xy)^{p+1}(x+y)^q + 4\binom{q}{2}(xy)^{p+2}(x+y)^{q-2}\right],$$

which yields that the γ-expansion of $G(A_{\mathbf{m}}(x,y))$ belongs to Type *II*. More precisely, there exist nonnegative integers $\widetilde{b}(m,k)$ such that

$$G\left(y\sum_{k=1}^{\lfloor m/2\rfloor} b(m,k)(xy)^k(x+y)^{m-2k}\right) = y\sum_{k=1}^{\lfloor (m+2)/2\rfloor} \widetilde{b}(m,k)(xy)^k(x+y)^{m+2-2k}. \tag{12}$$

(a_3) Consider $\mathbf{m} = \{m_1, m_2, \ldots, m_n\}$, where $\#\{m_i \in \mathbf{m} : m_i = 1\} = r$ and $\#\{m_i \in \mathbf{m} : m_i = 2\} = s$. Without loss of generality, assume that $1 \leqslant r, s < n$ and $r+s = n$. It follows from Lemma 7 and Theorem 8 that

$$A_{\mathbf{m}}(x,y) = G^s(T^r(x)).$$

Using (10), we see that there exist nonnegative integers $a(r,k)$ such that

$$G^s(T^r(x)) = G^s\left(\sum_{k=1}^{\lfloor (r+1)/2\rfloor} a(r,k)(xy)^k(x+y)^{r+1-2k}\right).$$

Repeatedly using (11) and (12), we find that there exist nonnegative integers $c(r,k)$ and $d(r,k)$ such that

$$A_{\mathbf{m}}(x,y) = \sum_{k\geqslant 1} c(r+2s,k)(xy)^k(x+y)^{r+2s+1-2k} + y\sum_{k\geqslant 1} d(r+2s,k)(xy)^k(x+y)^{r+2s-2k}.$$

So the γ-expansion of $A_{\mathbf{m}}(x,y)$ belongs to Type *III*. When $y=1$, we arrive at

$$A_{\mathbf{m}}(x) = \sum_{k\geqslant 1} c(r+2s,k)x^k(1+x)^{r+2s+1-2k} + \sum_{k\geqslant 1} d(r+2s,k)x^k(1+x)^{r+2s-2k},$$

as desired. This completes the proof. □

Example 10. Note that $A_{\{2\}}(x,y) = y(xy)$. Then $T\left(xy^2\right) = xy^2(2x+y) = (xy)^2 + y(xy)(x+y)$, and so the γ-expansion of $T\left(A_{\{2\}}(x,y)\right) = A_{\{2,1\}}(x,y)$ belongs to Type *III*. Moreover, we have

$$G\left(xy^2\right) = xy^2(x^2+4xy+y^2) = y\left((xy)(x+y)^2 + 2(xy)^2\right).$$

So the γ-expansion of $G\left(A_{\{2\}}(x,y)\right) = A_{\{2,2\}}(x,y)$ belongs to Type *II*.

From the proof of Theorem 4, we can conclude the following result.

Proposition 11. *Let $f(x,y)$ be a bivariate polynomial. If the γ-expansion of $f(x,y)$ is given by Type I, then the Eulerian operator T preserves the type of the γ-expansion of $f(x,y)$, while the γ-expansion of $G(f(x,y))$ belongs to Type III. If the γ-expansion of $f(x,y)$ is given by Type II, then the γ-expansion of $T(f(x,y))$ belongs to Type III, while the Eulerian operator G preserves the type of the γ-expansion of $f(x,y)$. If the γ-expansion of $f(x,y)$ is given by Type III, then both T and G preserve the type of the γ-expansion of $f(x,y)$.*

4. Concluding remarks

In this paper, we discover the commutative property of a pair of Eulerian operators. In recent years, the Eulerian polynomials have been extended to several combinatorial structures, including the types B and D Coxeter groups [8], weakly increasing trees [13], s-inversion sequences [17], Stirling permutations [20] and quasi-Stirling permutations [26]. We plan to explore some Eulerian operators in these combinatorial structures in future.

Data availability

No data was used for the research described in the article.

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